FOUNDATION YEAR 2025-2026

COURSE - DATA ANALYSIS

Lesson 5

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Poisson Distribution

Definition

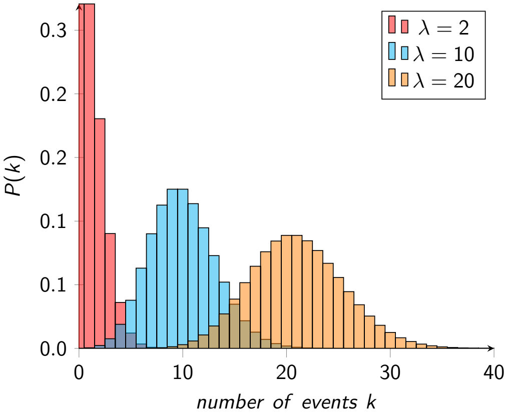
- The binomial distribution describes the occurring of k events out of n trials.
- The Poisson distribution describes cases in which there are still particular outcomes but no idea on the number of trials. The usual situation is occurrences of a certain number of events during a fixed time interval.
- The Poisson distribution is a limit of the binomial distribution with $n \to \infty$, $p \to 0$, but $n \cdot p = const$

Poisson PDF

$$P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \tag{1}$$

- $\lambda > 0$ is the average number of events
- k is the number of events
- e is Euler's number (e=2.71828...)
- ! is the factorial function

Poisson Distribution



- The **mean** of the Poisson distribution is $\mu = \lambda$.
- The **variance** of the Poisson distribution is $\sigma^2 = \lambda$.
- The **standard deviation** of the Poisson distribution is $\sigma = \sqrt{\lambda}$.

Poisson Distribution - example Fatal horse kicks 1/3

The number of fatal horse kicks recorded in 10 stables over 20 years is 122. Describe the distribution of fatal kicks per year for a stable using a Poisson model.
The mean number of kicks per year per stable is

$$\lambda = \frac{122\ kicks}{200\ stable \cdot year} = 0.61 \frac{kicks}{stable \cdot year}.$$

What are the probabilities for k kicks per stable per year ?

$$P(k=0, \lambda=0.61) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{0.61^0 e^{-0.61}}{0!} = 0.54$$

$$P(k=1, \lambda=0.61) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{0.61^1 e^{-0.61}}{1!} = 0.33$$

$$P(k=2, \lambda=0.61) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{0.61^2 e^{-0.61}}{2!} = 0.10$$

$$P(k=3, \lambda=0.61) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{0.61^3 e^{-0.61}}{3!} = 0.02$$

$$P(k=4, \lambda=0.61) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{0.61^3 e^{-0.61}}{4!} = 0.003$$

Poisson Distribution - example Fatal horse kicks 2/3

We can calculate $P(k = 5/6/.., \lambda = 0.61)$ but it would be very small. Remember that the total probability for different cases must sum to 1:

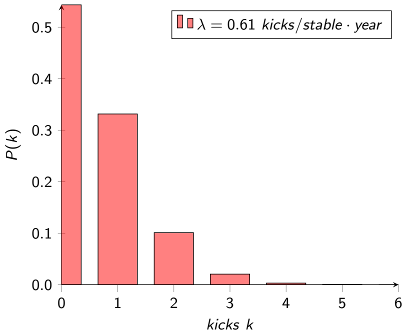
$$\sum_{k}^{\infty} P(k, \lambda) = 1$$

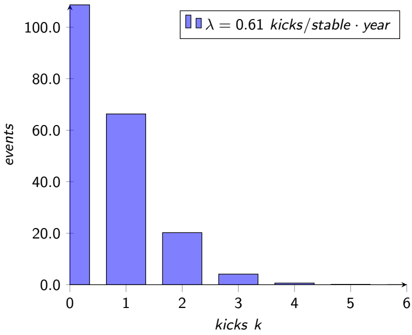
What is the expected number of k kicks in the 10 stables over the 20 year period ?

$$cases(0, n = 200) = P(0) \cdot 200 = 108.7$$

$$cases(1, n = 200) = P(1) \cdot 200 = 66.3$$
$$cases(2, n = 200) = P(2) \cdot 200 = 20.2$$
$$cases(3, n = 200) = P(3) \cdot 200 = 4.1$$
$$cases(4, n = 200) = P(4) \cdot 200 = 0.6$$

Poisson Distribution - example Fatal horse kicks 3/3





Number of kicks in 10 stable and 20 years	**Actual number of such cases**	**Poisson predictions**
0	109	108.7
1	65	66.3
2	22	20.2
3	3	4.1
4	1	0.6

Poisson Distribution - example Supernova Neutrinos

Here are the numbers of neutrino events detected in 10 second intervals by the Irvine-Michigan-Brookhaven experiment on 23 February 1987-around which time the supernova S 1987a was first seen by astronomers:

No. of neutrinos (k)	0	1	2	3	4	5	6	7	8	9
No. of intervals	1042	860	307	78	15	3	0	0	0	1
Prediction (n_k)	1064	823	318	82	16	2	0.3	0.03	0.003	0.0003

There are N=2306 intervals observed during the experiment.

$$\lambda = \frac{0 \cdot 1042 + 1 \cdot 860 + 2 \cdot 307 + 3 \cdot 78 + 4 \cdot 15 + 5 \cdot 3 + 6 \cdot 0 + 7 \cdot 0 + 8 \cdot 0 + 9 \cdot 1}{1042 + 860 + 307 + 78 + 15 + 3 + 0 + 0 + 0 + 1} = 0.77 \; neutrinos$$

What are the predicted number of intervals according to the Poisson distribution ? E.g.
$P(k=0,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{0.77^0 e^{-0.77}}{0!} = e^{-0.77} = 0.463$.
$n(k=0) = 2306 \cdot P(k=0,\lambda) = 2306 \cdot 0.463 = 1064$.
We can see that the Poisson predictions closely resemble the observations. These events are the background random events. The 9-event interval is a notable exception $\rightarrow$ SUPERNOVA.
Remember, in general $n_k = P(k) \cdot N$, N is the total number of observed events.

Example - Poisson and Binomial

The Poisson and Binomial distributions are very similar for large n and small p.

- $P(k, p, n) = p^k(1-p)^{n-k}\frac{n!}{k!(n-k)!}$ *Binomial*
- $P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$ *Poisson*

Example

For n=100 trials, with an individual probability of success p=2%, the Binomial and Poisson probabilities are:

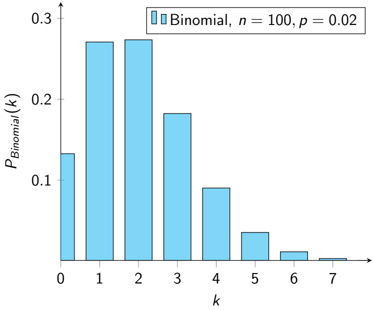
k	0	1	2	3	4	5	6
$P_{Binomial}$	13.3%	27.1%	27.3%	18.2%	9.0%	3.5%	1.1%
$P_{Poisson}$	13.5%	27.1%	27.1%	18.0%	9.0%	3.6%	1.2%

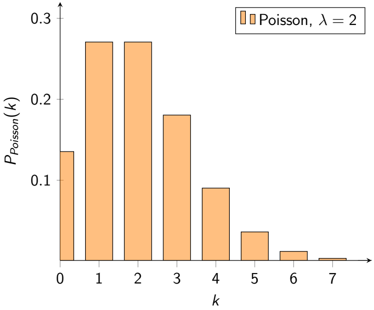
Let's calculate the two probabilities for k=0:

- $P_{Binomial}(k=0, p=0.02, 100) = 0.02^0(1-0.02)^{100-0}\frac{100!}{0!(100-0)!} = 0.98^{100} = 0.133$
- To evaluate the Poisson, first we must calculate $\lambda = 100 \cdot 0.02 = 2$.
  $P_{Poisson}(k=0, \lambda=2) = \frac{2^0 e^{-2}}{0!} = e^{-2} = 0.135$.

In the limit of large n and small p, you can approximate the Binomial with the Poisson distribution (easier to compute).

Example - Poisson and Binomial





What are the means, and σ of the two distributions ?

- Binomial: $\mu = n \cdot p = 100 \cdot 0.02 = 2$;
  $\sigma = \sqrt{np(1-p)} = \sqrt{100 \cdot 0.02 \cdot 0.98} = \sqrt{1.96} = 1.4$
- Poisson: $\lambda = \mu$; $\sigma = \sqrt{\lambda} = \sqrt{2} = 1.414$.

The Poisson distribution is always broader than the Binomial distribution with the same mean.

Exercise 1 - Binomial

A biologist examines frogs for a genetic trait he suspects might be due to contaminated water. Normally the trait examined presents itself on average in 1 out of 8 frogs in the wild. If the frequency of this trait has not changed and he captures 12 frogs to examine, what is the probability he finds this trait in more than 4 frogs? Given this, if he sees more than 4 frogs with the trait in question, what reasonable conclusion should he likely make?

Exercise 1 - Binomial

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Solution

- p = 1/8 = 0.125; n=12
- p(k>4) = p(k=5) + p(k=6) + p(k=7) + p(k=8) + ... + p(k=12) = 1 - [p(k=0) + p(k=1) + p(k=2) + p(k=3) + p(k=4)]
- $P_{Binomial}(k = 0, p = 1/8, 12) = 0.125^0(1 - 0.125)^{12-0}\frac{12!}{0!(12-0)!} = 0.98^{100} = 0.20142$
- $P_{Binomial}(k = 1, p = 1/8, 12) = 0.125^1(1 - 0.125)^{12-1}\frac{12!}{0!(12-1)!} = 0.34529$
- $P_{Binomial}(k = 2, p = 1/8, 12) = 0.125^2(1 - 0.125)^{12-2}\frac{12!}{0!(12-2)!} = 0.2713$
- $P_{Binomial}(k = 3, p = 1/8, 12) = 0.125^3(1 - 0.125)^{12-3}\frac{12!}{0!(12-3)!} = 0.12919$
- $P_{Binomial}(k = 4, p = 1/8, 12) = 0.125^4(1 - 0.125)^{12-4}\frac{12!}{0!(12-4)!} = 0.04153$

P(k>4) = 1 - (0.20142 + 0.34529 + 0.2713 + 0.12919 + 0.04153) = 0.01127

Exercise 2 - Binomial

Assume that 13% of people are left-handed. If we select 5 people at random, find the probability of each outcome below:

1. The first lefty is the fifth person chosen
2. There are exactly 3 lefties in the group
3. There are some lefties among the 5 people
4. There are no more than 3 lefties in the group

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1. $P(k=1, p=0.13, n=5)/5 = 0.13^1(1-0.13)^{5-1}\frac{5!}{1!4!} = 0.37238/5 = 0.074476$
2. $P(k=3, p=0.13, n=5) = 0.01663$
3. $P(k=1, p=0.13, n=5) + P(k=2, p=0.13, n=5) + P(k=3, p=0.13, n=5) + P(k=4, p=0.13, n=5) + P(k=5, p=0.13, n=5) = 1 - P(k=0, p=0.13, n=5) = 1 - 0.13^0(1-0.13)^{5-0}\frac{5!}{0!5!} = 1 - (0.87)^5 = 0.501579$
4. $P(k=0, p=0.13, n=5) + P(k=1, p=0.13, n=5) + P(k=2, p=0.13, n=5) + P(k=3, p=0.13, n=5) = 1 - [P(k=4, p=0.13, n=5) + P(k=5, p=0.13, n=5)] = 1 - [0.13^4(1-0.13)^{5-4}\frac{5!}{4!1!} + 0.13^5(1-0.13)^{5-5}\frac{5!}{5!0!}] = 1 - [0.00124 + 0.00004] = 0.99872$

Exercise 3 - Binomial

Is it unusual to see less than 3 heads in 12 flips of a coin? Why?

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Is it unusual to see less than 3 heads in 12 flips of a coin? Why? p(heads=0.5).
$P(k=0,0.5,12)+P(k=1,0.5,12)+P(k=2,0.5,12)=$
$0.5^0(1-0.5)^{12-0}\frac{12!}{0!12!}+0.5^1(1-0.5)^{12-1}\frac{12!}{1!11!}+0.5^2(1-0.5)^{12-2}\frac{12!}{2!10!}=0.01929.$
Yes, it only happens around 2% of the time.

Exercise 4 - Poisson

Initially there are 1,000,000 radioactive atoms of Cesium-137. After 365 days, 977,287 of these atoms remain undecayed. Use the Poisson distribution to estimate the probability that on a given day, 50 radioactive atoms decayed.

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Initially there are 1,000,000 radioactive atoms of Cesium-137. After 365 days, 977,287 of these atoms remain undecayed. Use the Poisson distribution to estimate the probability that on a given day, 50 radioactive atoms decayed.
$P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$
The average number of events in one day is: $\lambda = (1000000 - 977287)/365 = 62.227$.
$P(k = 50) = \frac{62.227^{50} e^{-62.227}}{50!} = 0.0155$

Exercise 5 - Poisson

In the last 100 years, there have been 93 earthquakes measuring 6.0 or more on the Richter scale. What is the probability of having more than 2 earthquakes in the same year that all measure 6.0 or more?

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$P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$

$\lambda = 93/100 = 0.93 earthquake/year.$

$P(k > 3) = 1 - [P(0) + P(1) + P(2)] = 1 - [\frac{0.93^0 e^{-0.93}}{0!} + \frac{0.93^1 e^{-0.93}}{1!} + \frac{0.93^2 e^{-0.93}}{2!}] =$
$1 - (0.39455 + 0.36694 + 0.17062) = 0.06789$

Exercise 6 - Poisson

Suppose the probability of suffering a fever from the flu vaccine is 0.005. If 1000 people are given the vaccine, use the Poisson distribution to approximate the probability that a) 1 person suffers a fever as a result; and b) more than 6 people suffer a fever as a result.

Exercise 6 - Poisson

Suppose the probability of suffering a fever from the flu vaccine is 0.005. If 1000 people are given the vaccine, use the Poisson distribution to approximate the probability that a) 1 person suffers a fever as a result; and b) more than 6 people suffer a fever as a result. $P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$

$\lambda = p * n = 0.005 * 1000 = 5$

a) $P(k = 1, \lambda = 5) = \frac{5^1 e^{-5}}{1!} = 0.03369$.

b) $1 - [P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6)] =$
$1 - [0.00674 + 0.03369 + 0.08422 + 0.14037 + 0.17547 + 0.17547 + 0.14622] = 0.23782$.